ON THE NUMBER OF DIRICHLET CHARACTERS WITH MODULUS NOT EXCEEDING x

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Let $\Psi(x)$ denote the number of primitive Dirichlet characters with modulus not exceeding x and define $\Phi(x) := \sum_{n \leqslant x} \varphi(n)$, where φ stands for Euler's totient function. In [2] we proved that

$$\lim_{x \to \infty} \frac{\Psi(x)}{\Phi(x)} = \frac{6}{\pi^2}.$$

We expressed this by saying that the proportion of primitive characters among the total number of characters, is $6\pi^{-2}$. Hence, by all characters with modulus not exceeding x we meant $\Phi(x)$. In doing so, a function $\chi: \mathbb{N} \to \mathbb{C}$ may be counted more than once in $\Phi(x)$. E.g., the function $\chi: \mathbb{N} \to \mathbb{C}$ which is periodic modulo 6 and for which $\chi(1)=1$, $\chi(5)=-1$, $\chi(2)=\chi(3)=\chi(4)=\chi(6)=0$ is counted twice in $\Phi(12)$, once as a character modulo 6 and again as a character modulo 12.

Looking at a Dirichlet character χ rather as a function $\chi: \mathbb{N} \to \mathbb{C}$ than as a function $\chi: M_n \to \mathbb{C}$, where M_n denotes the group of reduced residue classes modulo n, the number of characters with modulus not exceeding x is no longer $\Phi(x)$, but less.

In this note we investigate whether from this point of view the proportion of primitive characters among all characters is still $6\pi^{-2}$. The answer will turn out to be no, the proportion becomes $1.1344\ldots$ times greater.

Definition. Let χ denote a character of the group M_n of reduced residue classes modulo n. Suppose there exists a character χ' of $M_{n'}$ with $n' < n$, such that χ and χ', considered as functions $\mathbb{N} \to \mathbb{C}$, coincide. Then χ is called an *improper character of* M_n. A character χ of M_n which is not improper is called a *proper character of* M_n.

Some authors, see e.g. [3, p. 479], use the words proper and improper as alternatives for primitive and imprimitive, but here they have a different meaning.

A primitive character is always proper. In the above example χ is proper as a character of M_6 and improper as a character of M_{12}.

Our problem is now to give an asymptotic estimate for the number of proper characters with modulus not exceeding x.

In the sequel we shall use the arithmetical functions β, γ, I and ψ, defined as follows:

$\beta(n)=$the product of those primes that occur with multiplicity 1 in n,

$\gamma(n)=\prod_{p|n} p$, $\gamma(n)$ is called the *core* of n,

$I(n)=n$,

$\psi(n)=$the number of primitive characters of M_n.

In [2] we showed that ψ is multiplicative and that

$$\psi(p^k)=\varphi(p^k)-\varphi(p^{k-1}). \tag{1}$$

THEOREM 1. Let $\omega(n)$ denote the number of proper characters of M_n. Then ω is a multiplicative arithmetical function and

$$\omega(p)=\varphi(p),$$
$$\omega(p^k)=\varphi(p^k)-\varphi(p^{k-1}),\ k>1.$$

PROOF. A character χ of M_n is improper if and only if it is induced by a, not necessarily primitive, character χ' of $M_{n'}$ with $\gamma(n)=\gamma(n')$. Hence, a primitive character of M_d induces a proper character of M_n, $d|n$, when there does not exist a number k with $d|k$, $k|n$, $k\neq n$, $\gamma(k)=\gamma(n)$. From this we see, remembering moreover that primitive characters are proper, that

$$\omega(n)=\sum_{d|\beta(n)}\psi\left(\frac{n}{d}\right). \tag{2}$$

Let $(m, n)=1$. Then we have in view of the multiplicativity of β and ψ:

$$\omega(mn)=\sum_{d|\beta(mn)}\psi\left(\frac{mn}{d}\right)=\sum_{\substack{d_1|\beta(m)\\ d_2|\beta(n)}}\psi\left(\frac{m}{d_1}\right)\psi\left(\frac{n}{d_2}\right)=\omega(m)\,\omega(n).$$

Note that one can introduce a convolution $\circ$, where $h=f\circ g$ is defined by

$$h(n)=\sum_{d|\beta(n)}f(d)g\left(\frac{n}{d}\right).$$

This convolution conserves multiplicativity but is not commutative. It is an example of a type of convolutions introduced by NARKIEWICZ, see [4] or [1, p. 252].

Another proof of the multiplicativity of ω runs along the following lines. Let χ_1' be a character of M_m and χ_2' one of M_n. Let they induce respectively the characters χ_1 and χ_2 of M_{mn} and let χ of M_{mn} be defined by $\chi:=\chi_1\chi_2$. Every character χ of M_{mn} can be obtained in one and just one such a way. Now χ is proper, if and only if χ_1' and χ_2' are both proper.

The last statement of the theorem follows either directly from (1) and (2) or from (1) and the observation that all characters of M_p are proper and that the proper characters of M_{p^k}, $k>1$, are just the primitive ones.

All characters of M_n are proper, if and only if $\varphi(n)=\omega(n)$, that is when n is squarefree. Since $\omega(n)\neq 0$, $n=1, 2, \ldots$, there are no groups M_n whose characters are all improper. The notion of primitive and proper coincides for those M_n for which $\psi(n)=\omega(n)$. This is the case if and only if $\beta(n)=1$.

THEOREM 2. $\sum_{n\leqslant x}\omega(n)=\frac{1}{2}\varrho x^2+0(x\sqrt{x})$, $x\to\infty$, with

$$\varrho=\prod_p\left(1-\frac{p^2+p-1}{p^4}\right).$$

PROOF. Define the arithmetical function α by $\omega=\alpha\star I$, where $\star$ denotes the ordinary Dirichlet convolution. Then it is easily verified that

$$\left\{\begin{array}{l}\alpha(p)=-1,\\ \alpha(p^2)=-p+1,\\ \alpha(p^k)=0,\ k\geqslant 3.\end{array}\right.\tag{3}$$

From (3) and the multiplicativity of α it follows that

$$|\alpha(n)|\leqslant\sqrt{n},\ n=1, 2, \ldots.\tag{4}$$

Now we have, see [5, p. 12],

$$\sum_{n\leqslant x}\omega(n)=\frac{1}{2}\sum_{d=1}^{x}\alpha(d)\left\{\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)\right\}.$$

Using (4) it is easy to see that

$$\sum_{n\leqslant x}\omega(n)=\frac{1}{2}x^2\sum_{d=1}^{\infty}\frac{\alpha(d)}{d^2}+0(x\sqrt{x}),\ x\to\infty.$$

By (4) the infinite series converges absolutely. Writing it as an Euler product, using (3) and the multiplicativity of α, we see that its sum equals the constant ϱ from theorem 2.

Thus, if we look at a Dirichlet character as a function $\mathbb{N}\to\mathbb{C}$, i.e. if we consider only proper characters, the probability that a character be primitive is, see [2, (2)], $36\pi^{-4}\varrho^{-1}$. As it was $6\pi^{-2}$ in the old sense, it grew by a multiplicative factor $6\pi^{-2}\varrho^{-1}$, which equals

$$\prod_p\left(1+\frac{1}{p^3+p^2-1}\right)=1.1344\ \ldots\ .$$

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